Collapse of thin shell structures: Stress resultant plasticity modelling within a co-rotated ANDES finite element formulation

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SUMMARY

Due to the very nonlinear behaviour of thin shells under collapse, numerical simulations are subject to challenges. Shell finite elements are attractive in these simulations. Rotational degrees of freedom do, however, complicate the solution. In the present study a co-rotated formulation is employed. The deformation of the shell is decomposed in to a contribution from large rigid body rotation and a strain producing term. A triangular assumed strain shell finite element is used. Hence, a high performance elastic element is combined with the co-rotated formulation. In the co-rotated co-ordinate system the plasticity is accounted for by a simplifyed Ilyushin stress resultant yield surface. The stress update is determined from the backward Euler difference, and a consistent geometrical and material tangent stiffness is derived. Comparison with other published analysis results show that the present formulation gives acceptable accuracy.

Keywords

Large rotations, co-rotated formulation, stress resultant plasticity, assumed strain thin shell finite element

1. INTRODUCTION

Shell structures are frequently used in marine, mechanical, aerospace, and civil engineering applications due to efficient load carrying capability relative to material costs; aestethical reasons also may be a governing factor for using shells. Today there is a trend to utilise structural capacity as much as possible. This usually involves allowance for nonlinear material and geometrical behaviour. Two important cases where this is relevant are firstly reassessment of the capacity of existing structures reaching their original service life. Employing more advanced calculations of structural response, one often can show significant reserve strengths in the actual structure. This then supports the decision for prolonged service life. Secondly, assessment of both structural behaviour during damage scenarios, and residual (post-damage) strengths, typically requires calculations accounting for nonlinear behaviour. Such calculations in terms of shell structures are carried out by means of shell finite elements. In many cases the nonlinear deformations of the shell are governed by large rigid body translations and rotations, and moderate strain producing motions. Hence, one simplification in expressing strain is obtained by assuming negligible reduction in shell thickness. Further simplifications are zero stress in thickness direction and vanishing out-of-plane shear strain for thin shells (i.e. the structural planar dimensions are at least one order of magnitude larger than shell thickness). One complicating factor is the use of rotational degrees of freedom at the shell element nodes. They are not members of a linear space, hence for finite rotations special considerations are necessary. Many representations are possible (Euler parameters, shell director rotation, co-rotated formulation)[1−14]. In the present study the co-rotated formulation is employed. With this, the strain producing deformations and the (large) rigid body motion is split, and simplifications in strain description are easily carried out with respect to the co-rotated system. This also makes it possible to utilise high performance finite elements derived according to linear theory, hence taking advantage of the large efforts put into previous research on deriviation of finite elements with good convergence characteristics. Herein a triangular assumed natural deviatoric strain shell finite element (ANDES) presented by Felippa and co-workers is used[15,16]. This is a non-conforming element satisfying the individual element test by Bergan and Hanssen[17]. The material nonlinearity accounted for is due to plasticity. For shell elements two approaches are usual. Either a layer approach, i.e. integration over shell thickness, utilising a two-dimensional description of stress at each integration point. This approach may take advantage of the significant improvements over the last two decades in updating stress, notably by means of the backward Euler method. Alternatively, one may use a stress resultant approach. Then one avoids the integration over shell thickness, but the yield surfaces may become more complicated than in the layer approach. For instance, a layer approach with Mises material (no discontinuities in the yield surface gradient in the plane stress space) corresponds to a stress resultant yield surface with corners (Ilyushin). This may cause numerical problems. Remedies are given by Simo et al[18] (see also Ref.7), however, the implementation is more involved. In the present study this yield surface is simplified to a hyperellipse, avoiding the corners at the expense of introducing some inaccuracy in response calculation at inelastic integration points.

The main objectives with the present investigation are: 1) utilise a simplified plasticity theory (with obvious numerical advantages in stress resultant updating) 2) link this small strain plasticity description to a co-rotated formulation, and 3) investigate the performance of the ANDES shell element when utilised with plastic deformations[19,20]. The paper is organised as follows. First the shell kinematics is presented. This includes the co-rotated formulation and the deformational (strain producing) degrees of freedom. Secondly, the force equilibrium for an element is derived by means of variation of potential energy. Here, some of the transformations required for a correct equilibrium are presented. Then additional variation of the force equilibrium leads to the consistent tangent stiffness (that is used in the Newton-Raphson iterations with asymptotic quadratic rate of convergence). The stress resultant update and its linearization are then presented. Finally, several examples of shell problems are analysed and compared to other published simulations.

2. SHELL KINEMATICS

Fig.1 shows the two basic coordinate systems that are used. The global coordinate system is represented by unit vectors $\mathbf{I_1}$, $\mathbf{I_2}$ and $\mathbf{I_3}$. The co-rotated element coordinate system shared by shadow configuration C_{0n} and configuration C_n is represented by unit vectors $\mathbf{i_1^n}$, $\mathbf{i_2^n}$ and $\mathbf{i_3^n}$. The unit vectors describing the different coordinate systems are obtained as follows. $\mathbf{i_1^0}$ is aligned with edge 1-2 for the element in C_0 configuration. $\mathbf{i_3^0}$ (and $\mathbf{i_3^n}$) is always perpendicular to the shell element, and is obtained by the cross product of $\mathbf{i_1^0}$ and edge vector 1-3. Unit vector $\mathbf{i_2^0}$ is then $\mathbf{i_1^0}$x$\mathbf{i_3^0}$. The vector $\mathbf{i_1^n}$ and the shadow element C_{0n} is rotated an angle β relative to the edge 1-2 of the deformed element. β is defined by $\beta = -\frac{1}{3}(\alpha_1 + \alpha_2 + \alpha_3)$, and is the least square fit of the egde angular difference between C_{0n} and C_n. This makes the node numbering invariant of positioning of C_{0n} and C_n elements. Note that the present approach employs coinciding centroids for C_{0n} and C_n elements. Vectors given in the local co-rotated element coordinate system are marked with a tilde(˜). A vector $\mathbf{x}$ in global coordinates is transformed into a vector $\tilde{\mathbf{x}}$ in the local coordinate system *0* by

$$\tilde{\mathbf{x}} = \mathbf{T_0}\mathbf{x} \tag{1}$$

$$\mathbf{T_0} = \begin{bmatrix} \mathbf{i_1^0}^T \\ \mathbf{i_2^0}^T \\ \mathbf{i_3^0}^T \end{bmatrix} \tag{2}$$

$\mathbf{T_0}$ is orthonormal. The rigid body rotation of $\mathbf{i_i^0}$ to $\mathbf{i_i^n}$ is given by

$$\mathbf{i_1^n} = \mathbf{R_{0n}}\mathbf{i_1^0} \tag{3}$$

where $\mathbf{R_{0n}}$ is the rigid body rotation tensor from position *0* to position n. Utilising Eqn.(1) the rotation tensor reads

$$\mathbf{R_{0n}} = \mathbf{T_n^T}\mathbf{T_0} \tag{4}$$

2.1 Rotations in three-dimensional space

Large rotations in 2D commute. This is not the case for finite rotations in 3D. A number of different ways of treating large rotations in three dimensional space are possible, including rotation vector parameterization and orthogonal matrix parameterization (see for instance Ref.11). In this section, the Rodrigues representation of the rotation tensor is used. The rotation tensor for a rotation θ about an axis defined by the unit vector $\mathbf{n^T} = \begin{bmatrix} n_1 & n_2 & n_3 \end{bmatrix}$ is written[11] :

$$\mathbf{R} = \mathbf{I} + \mathbf{N}\sin\theta + \mathbf{N^2}\left(\mathbf{1} - \cos\theta\right) \tag{5}$$

$$\mathbf{N} = \mathbf{Spin}\left(\mathbf{n}\right) = \begin{bmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{bmatrix} \tag{6}$$

and $\mathbf{I}$ is the 3 by 3 identity matrix. Rotation of a vector $\mathbf{r_0}$ into $\mathbf{r}$ through an angle θ about an axis defined by the unit vector $\mathbf{n}$ is obtained by:

$$\mathbf{r} = \mathbf{R}\mathbf{r_0} \tag{7}$$

The explicit form of the rotation matrix reads

$$\mathbf{R}(\mathbf{n}, \theta) = \begin{bmatrix} 1 + (1-c)(n_1^2 - 1) & (1-c)n_1n_2 - n_3s & (1-c)n_1n_2 + n_2s \\ (1-c)n_2n_1 + n_3s & 1 + (1-c)(n_2^2 - 1) & (1-c)n_2n_3 - n_1s \\ (1-c)n_3n_1 - n_2s & (1-c)n_3n_2 + n_1s & 1 + (1-c)(n_3^2 - 1) \end{bmatrix} \tag{8}$$

where $c = \cos\theta$ and $s = \sin\theta$. Given $\mathbf{R}(\mathbf{n}, \theta)$, the rotation axis $\mathbf{n}$ and the rotation angle θ may be found from the following, in which indices (i, j, k) take on the cyclic permutations of (1, 2, 3):

$$d_i = n_i \sin\theta = \frac{1}{2}\left(R_{kj} - R_{jk}\right) \tag{9}$$

$\mathbf{n}$ is a unit vector, hence, $\sin\theta = \sqrt{d_1^2 + d_2^2 + d_3^2}$, and the rotation vector is found by $n_i = d_i / \sin\theta$. Thus the rotation vector associated with the rotation tensor is

$$\boldsymbol{\theta} = \theta\mathbf{n} = \frac{\theta}{\sin\theta}\begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \tag{10}$$

To avoid numerical difficulties, the fraction $\theta / \sin\theta$ is evaluated with a truncated Taylor series about $\theta = 0$ for small angles.

2.2 Translation of a point from C_0 configuration to C_n configuration.

Consider a body in initial configuration C_0 moving to configuration C_n. During the movement a point identified by subscript a is followed. The point is rigidly attached to another point through the eccentricity vector $\mathbf{e}_a$. In Fig.1 the different vectors are denoted by

$\mathbf{r_a^0}$	=	Position vector for node a in configuration C_0.
$\mathbf{r_a^n}$	=	Position vector for node a in configuration C_n.
$\mathbf{\acute{r}_a^0}$	=	Position vector for eccentricity node in configuration C_0.
$\mathbf{\acute{r}_a^n}$	=	Position vector for eccentricity node in configuration C_n.
$\mathbf{e}_a^0, \mathbf{e}_a^n$	=	Eccentricity vectors in configurations C_0 and C_n respectively.

Omitting a in the remaining equations of this section, the relationships between the vectors listed above are as follows:

$$\begin{aligned} \mathbf{r}^0 &= \acute{\mathbf{r}}^0 + \mathbf{e}^0 \\ \mathbf{r}^n &= \acute{\mathbf{r}}^n + \mathbf{e}^n = \acute{\mathbf{r}}^n + \mathbf{R}_{0n}\mathbf{e}^0 \end{aligned} \tag{11}$$

The displacement vector is given as the difference between the position vector in configuration C_0 and the position vector in configuration C_n.

$$\mathbf{u} = \mathbf{r^n} - \mathbf{r^0} = \acute{\mathbf{r^n}} + \mathbf{R_{0n}}\mathbf{e^0} - \acute{\mathbf{r^0}} - \mathbf{e^0} = \acute{\mathbf{u}} + (\mathbf{R_{0n}} - \mathbf{I})\,\mathbf{e^0} \tag{12}$$

The displacement vector is split into a deformational displacement vector and a rigid body displacement vector.

$$\mathbf{u} = \mathbf{u_r} + \mathbf{u_d} \tag{13}$$

Hence:

$$\mathbf{u_r} = \mathbf{r^{0n}} - \mathbf{r^0} \tag{14}$$

$$\mathbf{u_d} = \mathbf{r^n} - \mathbf{r^{0n}} \tag{15}$$

Introducing subscript c for the arithmetic mean of the coordinates of the points in the element, the position vectors in initial and shadow element configurations may be written as:

$$\mathbf{r^0} = \mathbf{r^0_c} + \mathbf{x^0} \tag{16}$$

$$\begin{aligned} \mathbf{r^{0n}} &= \mathbf{r^{0n}_c} + \mathbf{x^{0n}} \\ &= \mathbf{r^0_c} + \mathbf{u_c} + \mathbf{R_{0n}}\mathbf{x^0} \end{aligned} \tag{17}$$

where $\mathbf{x^0}$ and $\mathbf{x^{0n}}$ are the vectors from the centroid of the element to the point being considered in the C_0 configuration and the C_{0n} configuration respectively. Substitution of the expressions above into Eqs 14 and 15 yields:

$$\mathbf{u_d} = \mathbf{u} - \mathbf{u_r} = \mathbf{u} - (\mathbf{r^{0n}} - \mathbf{r^0}) = \mathbf{u} - (\mathbf{r^0_c} + \mathbf{u_c} + \mathbf{R_{0n}}\mathbf{x^0} - \mathbf{r^0_c} - \mathbf{x^0}) = \underline{\mathbf{u} - \mathbf{u_c} - (\mathbf{R_{0n}} - \mathbf{I})\mathbf{x^0}} \tag{18}$$

2.3 Rotation of an element node from configuration C_0 to C_n

The rotation of an element node as it moves from the initial configuration C_0 to the deformed configuration C_n is described by the rotation tensor $\mathbf{R}$. The rotation tensor is split into a rigid body rotation tensor $\mathbf{R_{0n}}$ and a deformational rotation tensor $\mathbf{R_d}$.

$$\mathbf{R} = \mathbf{R_d}\mathbf{R_{0n}} \tag{19}$$

$$\mathbf{R_d} = \mathbf{R}\mathbf{R^T_{0n}} = \mathbf{R}\mathbf{T^T_0}\mathbf{T_n} \tag{20}$$

The deformational rotation tensor transformed into the local coordinate system shared by configurations C_{0n} and C_n reads

$$\tilde{\mathbf{R}}_d = \mathbf{T_n}\mathbf{R_d}\mathbf{T^T_n} = \mathbf{T_n}\mathbf{R}\mathbf{T^T_0}\mathbf{T_n}\mathbf{T^T_n} = \mathbf{T_n}\mathbf{R}\mathbf{T^T_0} \tag{21}$$

2.4 Deformational displacement vector

The position of an element node a with initial coordinates $\acute{\mathbf{r}}_a^0$, is defined by the translational displacement $\mathbf{u_a}$ and the rotational orientation $\mathbf{R_a}$. Together, the set $(\mathbf{u_a}, \mathbf{R_a})$ for $a = 1, ..N$ is the nodal displacement vector $\hat{\mathbf{v}}$ "visible" to the other elements. $\hat{\mathbf{v}}$ is interpreted as an array of numbers that defines the position of the deformed element. In order to establish the strain energy, and thence the force vector and tangent stiffness for an element, the deformational vector for the element needs to be established. This vector is denoted $\tilde{\mathbf{v}}_d$ and contains translational and rotational degrees of freedom for each element node ordered as

$$\tilde{\mathbf{v}}_d^T = \left[\tilde{\mathbf{u}}_{d1}^T \tilde{\boldsymbol{\theta}}_{d1}^T \ldots \tilde{\mathbf{u}}_{dN}^T \tilde{\boldsymbol{\theta}}_{dN}^T \right] \tag{22}$$

N is the number of element nodes for the element being considered. $\tilde{\boldsymbol{\theta}}_d$ is extratced from $\tilde{\mathbf{R}}_\mathbf{d}$ in a similar manner as described by Eqs. 9 and 10.

3. POTENTIAL ENERGY AND FORCE EQUILIBRIUM

In the present section we consider elastic material behaviour. The plastic deformations are accounted for at element level in a later section. The potential energy for an element is

$$U_e = \int_V \int_{\tilde{\boldsymbol{\epsilon}}} \boldsymbol{\sigma}(\tilde{\boldsymbol{\epsilon}}) d\tilde{\boldsymbol{\epsilon}} dV \tag{23}$$

The first variation of the potential energy is

$$\delta\Pi = \delta(U_e + H) = \delta_R \tilde{\mathbf{v}}_d^T \tilde{\mathbf{f}}_e - \delta \mathbf{v}^T \mathbf{f}_{ext} = 0 \tag{24}$$

The task is to express the variation of co-rotated deformational DOF with respect to the global DOF

$$\delta_R \mathbf{v_d} = \frac{\partial_R \mathbf{v_d}}{\partial \mathbf{v}} \delta \mathbf{v} \tag{25}$$

The derivation leads to the following connection between internal and external element nodal forces

$$\mathbf{f}_e = \mathbf{E^T T^T \tilde{P}^T \tilde{H}^T \tilde{f}_e} = \mathbf{f_{ext}} \tag{26}$$

Some detail in this derivation is given in the following[19].

3.1 Variation of the deformational displacement

In the global coordinate system, the degrees of freedom at each node are $\mathbf{v^T} = \left[\begin{array}{cc} \mathbf{u^T} & \boldsymbol{\theta}^T \end{array} \right]^\mathbf{T}$. $\boldsymbol{\theta}$ is a representation of a rotation matrix defining the orientation of the node, while $\mathbf{u}$ contains the translations of the node. Thus, $\boldsymbol{\theta}$ is a representation of a finite three-dimensional rotation. On variation of $\mathbf{v}$, the *variation* of the rotations are no longer in the finite three-dimensional domain, but rather in the infinitesimal linear domain. Thus, the variation of the rotation matrix is represented by an infinitesimal rotation vector. The variatonal degrees of freedom are therefore $\delta\mathbf{v}^{\mathbf{T}} = \begin{bmatrix} \delta\mathbf{u}^{\mathbf{T}} & \delta\boldsymbol{\omega}^{T} \end{bmatrix}^{\mathbf{T}}$. Some of the variations are carried out with respect to the rigid body rotations. The finite rigid body rotations are denoted $\boldsymbol{\theta}_r$, but analogous with the argumentation above, the infinitesimal rigid body rotations are denoted $\delta\boldsymbol{\omega}_r$. Since the transformation matrix $\mathbf{T_n}$ transforms a vector from a global to a local coordinate system, and thus only "changes" the *orientation of the vector*, it depends of the variation of the rotation of the element. The translation and rotation considered are those applying to the centre of the element. The result is

$$\delta\mathbf{T_n} = \frac{\partial\mathbf{T_n}}{\partial\tilde{\omega}_\mathbf{i}}\delta\tilde{\omega}_\mathbf{i} = \begin{bmatrix} 0 & \delta\tilde{\omega}_z & -\delta\tilde{\omega}_y \\ -\delta\tilde{\omega}_z & 0 & \delta\tilde{\omega}_x \\ \delta\tilde{\omega}_y & -\delta\tilde{\omega}_x & 0 \end{bmatrix} \begin{bmatrix} \mathbf{i_1^{nT}} \\ \mathbf{i_2^{nT}} \\ \mathbf{i_3^{nT}} \end{bmatrix} = -\mathbf{Spin}\,(\delta\tilde{\boldsymbol{\omega}})\mathbf{T_n} \tag{27}$$

Transformation of $\mathbf{Spin}\,(\delta\tilde{\boldsymbol{\omega}})$ to global coordinates may be written as

$$\delta\mathbf{T_n} = -\mathbf{Spin}\,(\delta\tilde{\boldsymbol{\omega}})\mathbf{T_n} = -\mathbf{T_n}\mathbf{Spin}\,(\delta\boldsymbol{\omega})\mathbf{T_n}\mathbf{T_n^T} = -\mathbf{T_n}\mathbf{Spin}\,(\delta\boldsymbol{\omega}) \tag{28}$$

$\mathbf{Spin}\,(\delta\boldsymbol{\omega})$ is anti-symmetric.

The rotation tensor $\mathbf{R_{0n}}$ rotates a vector from initial configuration to the shadow configuration, hence the variation of the rotation tensor reads

$$\delta\mathbf{R_{0n}} = \delta\mathbf{T_n^T}\mathbf{T_0} + \mathbf{T_n^T}\delta\mathbf{T_0} = \delta\mathbf{T_n^T}\mathbf{T_0} = \mathbf{Spin}\,(\delta\boldsymbol{\omega})\mathbf{T_n^T}\mathbf{T_0} = \mathbf{Spin}\,(\delta\boldsymbol{\omega})\mathbf{R_{0n}} \tag{29}$$

The variation of a vector expressed in an inertial frame can be expressed as

$$\delta\mathbf{x} = \delta_\mathbf{R}\mathbf{x} + \delta\boldsymbol{\omega}\times\mathbf{x} = \delta_\mathbf{R}\mathbf{x} + \mathbf{Spin}\,(\delta\boldsymbol{\omega})\mathbf{x} \tag{30}$$

where $\delta_R\mathbf{x}$ is the variation of the vector in the co-rotated frame, and $\delta\boldsymbol{\omega}$ is the variation of the inertial rotation of the frame. The variation of the co-rotated deformational displacement vector $\mathbf{u_d}$ in a co-rotated frame is obtained via the variation of the inertial deformational displacements with respect to inertial $\mathbf{v}$, and using Eqn. 30 to find the variation of the co-rotated deformational displacements with respect to inertial $\mathbf{v}$. Eqn. 18 states that deformational displacement of an element node a is

$$\mathbf{u_{da}} = \mathbf{u_a} - \mathbf{u_c} - (\mathbf{R_{0n}} - \mathbf{I})\,\mathbf{x_a^0} = \sum_{\mathbf{b=1}}^{\mathbf{N}} \delta_\mathbf{ab}\mathbf{u_b} - \sum_{\mathbf{b=1}}^{\mathbf{N}} \frac{\mathbf{1}}{\mathbf{N}}\mathbf{u_b} - (\mathbf{R_{0n}} - \mathbf{I})\,\mathbf{x_a^0} \tag{31}$$

$\mathbf{u_c}$ is the displacement vector for the element centroid, and δ_{ab} is the Kronecker delta, hence

$$\begin{aligned} \mathbf{u_{da}} &= \textstyle\sum_{b=1}^{N} \mathbf{P_{ab}}\mathbf{u_b} - (\mathbf{R_{0n}} - \mathbf{I})\,\mathbf{x_a^0} \\ \mathbf{P_{ab}} &= \left(\delta_{ab} - \tfrac{1}{N}\right)\mathbf{I} \end{aligned} \tag{32}$$

The variation of inertial $\mathbf{u_{da}}$ with respect to inertial $\mathbf{v}$ is found as

$$\delta\mathbf{u_{da}} = \sum_{\mathbf{b=1}}^{\mathbf{N}} \mathbf{P_{ab}}\delta\mathbf{u_b} - \delta\mathbf{R_{0n}}\mathbf{x_a^0} \tag{33}$$

Using Eqn. 29 $\delta\mathbf{R_{0n}x_a^0}$ may be written as

$$\begin{aligned} \delta\mathbf{R_{0n}x_a^0} &= \mathbf{Spin}\,(\delta\boldsymbol{\omega}_\mathbf{r})\mathbf{R_{0n}x_a^0} = \mathbf{Spin}\,(\delta\boldsymbol{\omega}_\mathbf{r})\mathbf{x_a^{0n}} = -\mathbf{Spin}\,(\mathbf{x_a^{0n}})\delta\boldsymbol{\omega}_\mathbf{r} \\ &= -\mathbf{Spin}\,(\mathbf{x_a^{0n}})\mathbf{G}\delta\mathbf{v} \end{aligned} \tag{34}$$

The matrix $\mathbf{G}$ connects the variation of the rigid body rotation to the variation of the visible node displacements:

$$\delta\boldsymbol{\omega}_r = \frac{\partial\boldsymbol{\omega}_r}{\partial v_i}\delta v_i = \mathbf{G}\delta\mathbf{v} = \sum_{\mathbf{b=1}}^{\mathbf{N}} \mathbf{G_b}\delta\mathbf{v_b} \tag{35}$$

The matrix $\mathbf{G}$ is an element-*type* dependent matrix. The variation of inertial $\mathbf{u_d}$ with respect to inertial $\mathbf{v}$ then reads

$$\delta\mathbf{u_{da}} = \sum_{\mathbf{b=1}}^{\mathbf{N}} \left(\begin{bmatrix} \mathbf{P_{ab}} & \mathbf{0} \end{bmatrix} + \mathbf{Spin}\left(\mathbf{x_a^{0n}}\right)\mathbf{G_b}\right)\delta\mathbf{v_b} \tag{36}$$

$\delta\mathbf{v_b}$ is the inertial degrees of freedom for node b. Using the relationship $\mathbf{u_{da}} = \mathbf{x_a^n} - \mathbf{x_a^{0n}}$, we find that the variation of the co-rotated deformational displacement vector with respect to inertial degrees of freedom is given by

$$\delta_R\mathbf{u_{da}} = \delta_\mathbf{R}\mathbf{x_a^n} - \delta_\mathbf{R}\mathbf{x_a^{0n}} = \delta_\mathbf{R}\mathbf{x_a^n} \tag{37}$$

Since $\mathbf{x_a^n} = \mathbf{R_{0n}x_a^0} + \mathbf{u_{da}}$, the variation of inertial $\mathbf{x_a^n}$ with respect to inertial degrees of freedom is given by

$$\delta\mathbf{x_a^n} = \delta\mathbf{R_{0n}x_a^0} + \mathbf{R_{0n}}\delta\mathbf{x_a^0} + \delta\mathbf{u_{da}} = \sum_{\mathbf{b=1}}^{\mathbf{N}} \mathbf{P_{ab}}\delta\mathbf{u_b} \tag{38}$$

Substituting $\mathbf{x_a^n}$ for $\mathbf{x}$ in Eqn. 30 and solving with respect to $\delta_R\mathbf{x_a^n}$ yields

$$\begin{aligned} \delta_R\mathbf{x_a^n} &= \textstyle\sum_{b=1}^{N} \mathbf{P_{ab}}\delta\mathbf{u_b} - \mathbf{Spin}\,(\delta\boldsymbol{\omega}_\mathbf{r})\mathbf{x_a^n} = \sum_{\mathbf{b=1}}^{\mathbf{N}} \mathbf{P_{ab}}\delta\mathbf{u_b} + \mathbf{Spin}\,(\mathbf{x_a^n})\delta\boldsymbol{\omega}_\mathbf{r} \\ &= \textstyle\sum_{b=1}^{N} \left(\begin{bmatrix} \mathbf{P_{ab}} & \mathbf{0} \end{bmatrix} + \mathbf{Spin}\,(\mathbf{x_a^n})\mathbf{G_b}\right)\delta\mathbf{v_b} \end{aligned} \tag{39}$$

Hence, we have the variation of co-rotated deformational displacement vector with respect to inertial degrees of freedom $\delta_R\mathbf{u_{da}} = \delta_\mathbf{R}\mathbf{x_a^n}$.

As was the case for co-rotated deformational displacement, we can not find the variation of the co-rotated deformational rotations with respect to inertial degrees of freedom directly. However, the variation of the co-rotated deformational (finite) rotations with respect to the co-rotated deformational (infinitesimal) rotations was obtained by Nour-Omid and Rankin[9] based on a relationship established by Simo[13] and Szwabowicz[4]:

$$\delta_R\boldsymbol{\theta}_{da} = \frac{\partial\boldsymbol{\theta}_{da}}{\partial\boldsymbol{\omega}_{da}}\delta_R\boldsymbol{\omega}_{da} = \frac{\partial\left(\mathbf{Axial}\,(\ln(\mathbf{R_{da}}))\right)}{\partial\boldsymbol{\omega}_{da}}\delta_R\boldsymbol{\omega}_{da} = \mathbf{H_a}\delta_\mathbf{R}\boldsymbol{\omega}_\mathbf{da} \tag{40}$$

The matrix $\mathbf{H_a}$ is defined as

$$\mathbf{H_a} = \frac{\partial \boldsymbol{\theta}_\mathbf{a}}{\partial \boldsymbol{\omega}} = \mathbf{I} - \frac{\mathbf{1}}{\mathbf{2}}\mathbf{Spin}\left(\boldsymbol{\theta}_\mathbf{a}\right) + \eta \mathbf{Spin}\left(\boldsymbol{\theta}_\mathbf{a}\right)^\mathbf{2} \tag{41}$$

where

$$\eta = \frac{\sin(\frac{1}{2}\theta_a - \frac{1}{2}\theta_a \cos(\frac{1}{2}\theta_a)}{\theta_a^2 \sin(\frac{1}{2}\theta_a)} \qquad \text{and} \qquad \theta_a = \sqrt{\boldsymbol{\theta}_a^T \boldsymbol{\theta}_a} = \|\theta_b\| \tag{42}$$

To avoid numerical problems, η is computed from a truncated power series for small angles. The variation of the co-rotated deformational rotation $\delta_R \boldsymbol{\omega}_{da}$ with respect to inertial degrees of freedom is the difference between variation of nodal rotation $\delta \boldsymbol{\omega}_a$ and of rigid body rotation $\delta \boldsymbol{\omega}_r$, both varied with respect to inertial degrees of freedom.

$$\delta_R \boldsymbol{\omega}_{da} = \delta \boldsymbol{\omega}_a - \delta \boldsymbol{\omega}_r = \delta \boldsymbol{\omega}_a - \frac{\partial \boldsymbol{\omega}_r}{\partial v_i} \delta v_i = \delta \boldsymbol{\omega}_a - \mathbf{G_a} \delta \mathbf{v_a} \tag{43}$$

$\mathbf{G}$ is defined in Eqn. 35. $\delta_R \boldsymbol{\omega}_{da}$ may now be written as

$$\delta_R \boldsymbol{\omega}_{da} = \sum_{b=1}^{N} \left(\delta_{ab} \begin{bmatrix} \mathbf{0} & \mathbf{I} \end{bmatrix} - \mathbf{G_b} \right) \delta \mathbf{v_b} \tag{44}$$

Introducing Eqn. 44 into Eqn. 40 yields the final expression for the variation of co-rotated deformational rotation with respect to inertial degrees of freedom.

$$\delta_R \boldsymbol{\theta}_{da} = \mathbf{H_a} \sum_{\mathbf{b=1}}^{\mathbf{N}} \left(\delta_\mathbf{ab} \begin{bmatrix} \mathbf{0} & \mathbf{I} \end{bmatrix} - \mathbf{G_b} \right) \delta \mathbf{v_b} \tag{45}$$

For an element with N nodes, the nodal degrees of freedom are ordered as follows

$$\mathbf{v^T} = \begin{bmatrix} \mathbf{u_1^T} & \boldsymbol{\theta}_1^T & \ldots & \mathbf{u_N^T} & \boldsymbol{\theta}_N^T \end{bmatrix} \tag{46}$$

If Eqs. 39 and 45 are ordered accordingly, the $\delta_R \mathbf{v_d}$ may be written as

$$\delta_R \mathbf{v_d} = \mathbf{H}\left(\mathbf{I} - \mathbf{P_T} - \mathbf{P_R}\right) \delta \mathbf{v} = \mathbf{HP} \delta \mathbf{v} \tag{47}$$

Finally $\mathbf{I} - \mathbf{P_T} - \mathbf{P_R}$ is abbreviated into $\mathbf{P}$. Matrix $\mathbf{P}$ is a nonlinear projector operator. The details of these matrices are given in Appendix 1.

In order to account for nodal eccentricities, the eccentric degrees of freedom $\acute{\mathbf{v}}_a$ are used instead of the degrees of freedom $\mathbf{v_a}$. The relationship between $\delta \acute{\mathbf{v}}_a$ and $\delta \mathbf{v_a}$ is

$$\delta \mathbf{v_a} = \mathbf{E_a} \delta \acute{\mathbf{v}}_\mathbf{a} \qquad\qquad \mathbf{E_a} = \begin{bmatrix} \mathbf{I} & -\mathbf{Spin}\left(\mathbf{e_a^n}\right) \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \tag{48}$$

Observe that for all nodes with eccentricity vector $\mathbf{e}_a^n = \mathbf{0}$, $\mathbf{E_a}$ reduces to the identity matrix, thus leaving the elastic degrees of freedom untouched for node a. For an element with N eccentric nodes, this relationship is expands to

$$\delta \mathbf{v} = \begin{bmatrix} \mathbf{E_1} & \mathbf{0} & \ldots & \mathbf{0} \\ \mathbf{0} & \mathbf{E_2} & \ldots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \ldots & \mathbf{E_N} \end{bmatrix} \delta \acute{\mathbf{v}} = \mathbf{E} \delta \acute{\mathbf{v}} \tag{49}$$

Introducing Eqn. 49 into Eqn. 47 finally yields the expression for the variation of co-rotated deformational displacement vector with respect to inertial, eccentric, degrees of freedom for an element with N nodes:

$$\delta_R \mathbf{v_d} = \mathbf{HPE}\delta\acute{\mathbf{v}} \tag{50}$$

The variation of the co-rotated deformational displacement vector has now been found with respect to a set of inertial degrees of freedom. The choice of inertial system has, however, not been specified. To enable use of already existing linear elements put in the rotating frame, the inertial system is chosen to be that of the co-rotated element. Thus, Eqn. 47 is modified to

$$\delta_R \tilde{\mathbf{v}}_d = \tilde{\mathbf{H}}\tilde{\mathbf{P}}\delta\tilde{\mathbf{v}} \tag{51}$$

Since the visible degrees of freedom are defined in the global coordinate system, the variation needed is $\delta_R \mathbf{v_d}$ with respect to $\delta\mathbf{v}$. Taking advantage of the transformation between local and global coordinate system, and keeping the eccentricity matrix $\mathbf{E}$ in global coordinates, Eqn. 50 may finally be written as (see also Eqn.26)

$$\delta_R \tilde{\mathbf{v}}_d = \tilde{\mathbf{H}}\tilde{\mathbf{P}}\mathbf{T_n}\delta\mathbf{v} = \tilde{\mathbf{H}}\tilde{\mathbf{P}}\mathbf{TE}\delta\acute{\mathbf{v}} \tag{52}$$

4. CONSISTENT TANGENT STIFFNESS

The consistent tangent stiffness is defined by the variation of the internal force vector $\mathbf{f}_e$ with respect to the visible degrees of freedom, $\acute{\mathbf{v}}$:

$$\delta\mathbf{f}_e = \frac{\partial\mathbf{f}}{\partial\acute{\mathbf{v}}}\delta\acute{\mathbf{v}} = \mathbf{K_t}\delta\acute{\mathbf{v}} \tag{53}$$

Using Eqn. 26 for $\mathbf{f}_e$, Eqn. 53 yields:

$$\begin{aligned}\delta\mathbf{f} \;=\; & \delta\mathbf{E^T}\mathbf{T^T}\tilde{\mathbf{P}}^\mathbf{T}\tilde{\mathbf{H}}^\mathbf{T}\tilde{\mathbf{f}}_\mathbf{e} + \mathbf{E^T}\delta\mathbf{T^T}\tilde{\mathbf{P}}^\mathbf{T}\tilde{\mathbf{H}}^\mathbf{T}\tilde{\mathbf{f}}_\mathbf{e} + \mathbf{E^T}\mathbf{T^T}\delta_\mathbf{R}\tilde{\mathbf{P}}^\mathbf{T}\tilde{\mathbf{H}}^\mathbf{T}\tilde{\mathbf{f}}_\mathbf{e} + \mathbf{E^T}\mathbf{T^T}\tilde{\mathbf{P}}^\mathbf{T}\delta_\mathbf{R}\tilde{\mathbf{H}}^\mathbf{T}\tilde{\mathbf{f}}_\mathbf{e} \\ & + \mathbf{E^T}\mathbf{T^T}\tilde{\mathbf{P}}^\mathbf{T}\tilde{\mathbf{H}}^\mathbf{T}\delta\tilde{\mathbf{f}}_\mathbf{e} = (\mathbf{K_{GE}} + \mathbf{K_{GR}} + \mathbf{K_{GP}} + \mathbf{K_{GM}} + \mathbf{K_{MG}})\,\delta\acute{\mathbf{v}} = \mathbf{K_T}\delta\acute{\mathbf{v}}\end{aligned} \tag{54}$$

The different terms of the tangent stiffness represent eccentricity geometric stiffness, rotational geometric stiffness, projection geometric stiffness, moment correction geometric stiffness, and material and internal geometric stiffness, respectively. Some of the contributions to the tangent stiffness will be explained briefly in the following.

The eccentricity geometric stiffness arises from variation of the eccentricity matrix $\mathbf{E}$, and relates the changes in the internal force vector due to changes in the eccentric degrees of freedom. Details may be found in Ref.21.

The rotational geometric stiffness arises from the variation of the transformation matrix between initial configuration C_0 and shadow configuration C_n, and reflects the variation in the force vector with respect to the rigid body rotation of the element. A rigid rotation of a stressed element obviously rotates the stresses, in turn causing the internal forces to change direction to preserve equilibrium. Contracted with the local projected internal force vector $\tilde{\mathbf{f}} = \tilde{\mathbf{P}}^T\tilde{\mathbf{H}}^T\tilde{\mathbf{f}}_e$, where $\tilde{\mathbf{f}}$ contains pairs of internal forces and moments for each node ordered nodewise, the rotational geometric stiffness may be found from

$$\begin{aligned}\mathbf{E^T}\delta\mathbf{T^T}\tilde{\mathbf{P}}^\mathbf{T}\tilde{\mathbf{H}}^\mathbf{T}\tilde{\mathbf{f}}_\mathbf{e} &= \mathbf{E^T}\delta\mathbf{T^T}\tilde{\mathbf{f}} = \mathbf{E^T}\begin{bmatrix} \delta\mathbf{T_n^T} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \delta\mathbf{T_n^T} & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \delta\mathbf{T_n^T} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \delta\mathbf{T_n^T} \end{bmatrix}\begin{bmatrix} \tilde{\mathbf{n}}_1 \\ \tilde{\mathbf{m}}_1 \\ \vdots \\ \tilde{\mathbf{n}}_N \\ \tilde{\mathbf{m}}_N \end{bmatrix} \\ &= -\mathbf{E^T}\mathbf{T^T}\tilde{\mathbf{F}}_\mathbf{nm}\delta\tilde{\boldsymbol{\omega}}_\mathbf{r} = -\mathbf{E^T}\mathbf{T^T}\tilde{\mathbf{F}}_\mathbf{nm}\tilde{\mathbf{G}}\mathbf{T}\mathbf{E}\delta\acute{\mathbf{v}} = \mathbf{K_{GR}}\delta\acute{\mathbf{v}}\end{aligned} \tag{55}$$

See Appendix 2 for details.

The equilibrium projection geometric stiffness arises from the variation of the projector matrix $\tilde{\mathbf{P}}^T$, and reflects the variation of the force vector due to variations in the degrees of freedom. By decomposing the force vector $\tilde{\mathbf{H}}^T\tilde{\mathbf{f}}_e$ into unbalanced forces $\tilde{\mathbf{f}}_u = (\mathbf{I} - \tilde{\mathbf{P}}^\mathbf{T})\tilde{\mathbf{H}}^\mathbf{T}\tilde{\mathbf{f}}_\mathbf{e}$ and balanced forces $\tilde{\mathbf{f}}_b = \tilde{\mathbf{P}}^T\tilde{\mathbf{H}}^T\tilde{\mathbf{f}}_e$ (that is $\tilde{\mathbf{H}}^T\tilde{\mathbf{f}}_e = \tilde{\mathbf{f}}_u + \tilde{\mathbf{f}}_b$), the equilibrium projection geometric term in Eqn. 54 may be written as

$$\mathbf{E^T}\mathbf{T^T}\delta_\mathbf{R}\tilde{\mathbf{P}}^\mathbf{T}\tilde{\mathbf{H}}^\mathbf{T}\tilde{\mathbf{f}}_\mathbf{e} = -\mathbf{E^T}\mathbf{T^T}\left(\tilde{\mathbf{G}}^\mathbf{T}\delta_\mathbf{R}\tilde{\mathbf{S}}^\mathbf{T}\tilde{\mathbf{f}}_\mathbf{b} + \delta_\mathbf{R}\tilde{\mathbf{P}}^\mathbf{T}\tilde{\mathbf{f}}_\mathbf{u}\right) \tag{56}$$

$\tilde{\mathbf{S}}^T$ represents the rigid body rotation vectors (see Eqn.84in App.2), causing $\tilde{\mathbf{S}}^T\tilde{\mathbf{f}}_b = \mathbf{0}$, because balanced forces do not produce any work on a structure during rigid body displacement or rotation. Furthermore, $\delta\tilde{\mathbf{P}}^T\tilde{\mathbf{f}}_u$ can be neglected because it will be very small when C_{0n} and C_n are close. Eqn. 56 reduces to

$$\mathbf{E^T}\mathbf{T^T}\delta\tilde{\mathbf{P}}^\mathbf{T}\tilde{\mathbf{H}}^\mathbf{T}\tilde{\mathbf{f}}_\mathbf{e} = -\mathbf{E^T}\mathbf{T^T}\tilde{\mathbf{G}}^\mathbf{T}\tilde{\mathbf{F}}_\mathbf{n}^\mathbf{T}\delta_\mathbf{R}\tilde{\mathbf{v}}_\mathbf{d} = -\mathbf{E^T}\mathbf{T^T}\tilde{\mathbf{G}}^\mathbf{T}\tilde{\mathbf{F}}_\mathbf{n}^\mathbf{T}\tilde{\mathbf{P}}\mathbf{T}\mathbf{E}\delta\acute{\mathbf{v}} = \mathbf{K_{GP}}\delta\acute{\mathbf{v}} \tag{57}$$

where

$$\tilde{\mathbf{F}}_n = \begin{bmatrix} \mathbf{Spin}\left(\tilde{\mathbf{n}}_\mathbf{1}\right) \\ \mathbf{0} \\ \vdots \\ \mathbf{Spin}\left(\tilde{\mathbf{n}}_\mathbf{N}\right) \\ \mathbf{0} \end{bmatrix} \tag{58}$$

The moment correction geometric stiffness arises from variation of the rotation pseudo-vector Jacobian $\tilde{\mathbf{H}}$. Splitting the internal force vector into translational internal forces $\tilde{\mathbf{n}}$ and rotational internal moments $\tilde{\mathbf{m}}$, the moment correction term in Eqn. 54 may be written as:

$$\mathbf{E^T}\mathbf{T^T}\tilde{\mathbf{P}}^\mathbf{T}\delta_\mathbf{R}\tilde{\mathbf{H}}^\mathbf{T}\tilde{\mathbf{f}}_\mathbf{e} = \mathbf{E^T}\mathbf{T^T}\tilde{\mathbf{P}}^\mathbf{T}\tilde{\mathbf{M}}\tilde{\mathbf{P}}\mathbf{T}\mathbf{E}\delta\acute{\mathbf{v}} = \mathbf{K_{GM}}\delta\acute{\mathbf{v}} \tag{59}$$

where

$$\tilde{\mathbf{M}} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \tilde{\mathbf{M}}_1 & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \tilde{\mathbf{M}}_N \end{bmatrix} \tag{60}$$

and $\tilde{\mathbf{M}}_a$ is defined from the relationship[9]

$$\delta\tilde{\mathbf{H}}_a^T\tilde{\mathbf{m}}_a = \frac{\partial\tilde{\mathbf{H}}_a^T}{\partial\tilde{\boldsymbol{\omega}}_a}\tilde{\mathbf{m}}_a\delta\tilde{\boldsymbol{\omega}}_a = \tilde{\mathbf{M}}_a\delta\tilde{\boldsymbol{\omega}}_a. \tag{61}$$

The following expression for the consistent tangent stiffness is determined:

$$\begin{aligned} \mathbf{K_t} &= \mathbf{K_{GE}} + \mathbf{E^T T^T}\left(\tilde{\mathbf{K}}_{\mathbf{MG}} + \tilde{\mathbf{K}}_{\mathbf{GM}} + \tilde{\mathbf{K}}_{\mathbf{GR}} + \tilde{\mathbf{K}}_{\mathbf{GP}}\right)\mathbf{TE} \\ &= \mathbf{K_{GE}} + \mathbf{E^T T^T}\left(\tilde{\mathbf{P}}^{\mathbf{T}}\tilde{\mathbf{H}}^{\mathbf{T}}\tilde{\mathbf{K}}_{\mathbf{e}}\tilde{\mathbf{H}}\tilde{\mathbf{P}} + \tilde{\mathbf{P}}^{\mathbf{T}}\tilde{\mathbf{M}}\tilde{\mathbf{P}} - \tilde{\mathbf{F}}_{\mathbf{nm}}\tilde{\mathbf{G}} - \tilde{\mathbf{G}}^{\mathbf{T}}\tilde{\mathbf{F}}_{\mathbf{n}}^{\mathbf{T}}\tilde{\mathbf{P}}\right)\mathbf{TE} \end{aligned} \tag{62}$$

$\tilde{\mathbf{K}}_e$ represents the material stiffness and may include plasticity effects. It connects the local deformational dof increment with the local force increment:

$$\delta\tilde{\mathbf{f}}_e = \tilde{\mathbf{K}}_e\delta_R\tilde{\mathbf{v}}_d \tag{63}$$

and is detailled in the next section.

5. PLASTICITY FORMULATION

The layer approach with integration of a two-dimensional stress state over shell thickness is well established. Typical number of integration points is 5-7. This captures first fibre yielding. A stress resultants approach avoids this integration. The stress resultant yield condition derived by Ilyushin may be written

$$\begin{aligned} f(\tilde{\boldsymbol{n}},\tilde{\boldsymbol{m}}) &= (\frac{\bar{N}}{t^2} + \frac{4s\bar{P}}{\sqrt{3}t^3} + \frac{16\bar{M}}{t^4})^{0.5} - \sigma_0 = 0 \\ \bar{N} &= N_x^2 + N_y^2 - N_xN_y + 3N_{xy}^2 \\ \bar{M} &= M_x^2 + M_y^2 - M_xM_y + 3M_{xy}^2 \\ \bar{P} &= N_xM_x + N_yM_y - 0.5N_xM_y - 0.5N_yM_x + 3N_{xy}M_{xy} \\ s &= P/abs(P) = \pm 1 \end{aligned} \tag{64}$$

For thin shells of Mises material this criterion is quite good. Denoting the integration point stress resultant vector by $\boldsymbol{\sigma} = [\tilde{\boldsymbol{n}},\tilde{\boldsymbol{m}}]^T$ the yield criterion is rewritten in quadratic form

$$f = \boldsymbol{\sigma}^T\boldsymbol{A}\boldsymbol{\sigma} - (1 + \frac{H\epsilon_p}{\sigma_y})^2 = 0 \tag{65}$$

$$
\begin{aligned}
\boldsymbol{A} &= \begin{bmatrix} \frac{1}{n_0^2}\boldsymbol{I} & \frac{s}{2\sqrt{3}m_0 n_0}\boldsymbol{I} \\ \frac{s}{2\sqrt{3}m_0 n_0}\boldsymbol{I} & \frac{1}{m_0^2}\boldsymbol{I} \end{bmatrix} \bar{\boldsymbol{A}} \\
\bar{\boldsymbol{A}} &= \begin{bmatrix} 1 & -0.5 & 0 \\ -0.5 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \\
m_0 &= 0.25\sigma_y t^2, n_0 = \sigma_y t
\end{aligned}
$$

This form is employed with success by Ibrahimbegovic and Frey[22], and is based on work by Matthies[23]. The off-diagonal submatrices in $\boldsymbol{A}$ leads to corners in the yield surface, see Fig.2. If one assumes $s = 0$, a hyperellipse is obtained as yield surface. This leads to nonconservative inaccuracies as shown in Fig.2. The maximum error is approximately 12% for balanced membrane and bending loading. Approaching each axis, the error vanish. Note that there already is introduced some simplification in Ilyushin criterion. In the following s=0 is employed. Denoting the incremental strains (membrane and curvature terms) conjugate to the shell stress resultants by $\Delta\boldsymbol{\epsilon} = [\Delta\boldsymbol{\epsilon}_m, \Delta\boldsymbol{\kappa}]^T$, and utilising an associated flow rule, the backward Euler (BE) update of the plastic strain increment reads

$$
\Delta\boldsymbol{\epsilon}_{p,n+1} = \Delta\lambda_{n+1}\frac{\partial f}{\partial\boldsymbol{\sigma}}_{n+1} \tag{66}
$$

Here $n+1$ corresponds to the current load step in the global Newton-Raphson equilibrium iteration. A work hardening model is used, leading to the following connection between stress resultant quantities and equivalent scalar quantities:

$$
\boldsymbol{\sigma}^T d\boldsymbol{\epsilon} = \bar{\sigma} d\epsilon_p \Rightarrow d\epsilon_p = 2\bar{\sigma} d\lambda \qquad \bar{\sigma} = \sqrt{\boldsymbol{\sigma}^T \boldsymbol{A}\boldsymbol{\sigma}} \tag{67}
$$

Note that this $d\epsilon_p$ is not represented by the uniaxial plastic strain increment in a tensile test. Hence, the hardening modulus H employed is not equal to the slope of the uniaxial stress-strain curve. One advantage with the simplified yield surface is that one only have one active yield surface at any time in the stress update. Simo and Kennedy along with Peng and Crisfield have, however, employed the Ilyushin-Shapiro yield surface with the possibility of two active surfaces[6,7]. In the elastic predictor plastic corrector BE approach applied herein, the stress update is obtained by

$$
\begin{aligned}
\boldsymbol{\sigma}_{n+1} &= \boldsymbol{\sigma}_{trial} - \boldsymbol{C}\Delta\boldsymbol{\epsilon}_{p,n+1} \qquad (68) \\
\boldsymbol{\sigma}_{n+1} &= \bar{\boldsymbol{Q}}^{-1}\boldsymbol{\sigma}_{trial} \qquad \bar{\boldsymbol{Q}} = [\boldsymbol{I} + 2\Delta\lambda\boldsymbol{C}\boldsymbol{A}] \\
\boldsymbol{C} &= \begin{bmatrix} t\boldsymbol{D} & 0 \\ 0 & \frac{t^3}{12}\boldsymbol{D} \end{bmatrix} \\
\boldsymbol{D} &= \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}
\end{aligned}
$$

As $\boldsymbol{\sigma}_{n+1}$ now only depends on $\Delta\lambda$, the discrete yield condition f_{n+1} also does. Solving for $f(\Delta\lambda_{n+1})$ the stress update is directly obtained from Eqn.68. Solution of the nonlinear equation $f(\Delta\lambda_{n+1}) = 0$ by Newton-Raphson iterations is simplified by re-expressing $\bar{\boldsymbol{Q}}^{-1}$ explicitly by means of an eigenvector matrix $\boldsymbol{E}$ and eigenvalue matrix $\boldsymbol{\Gamma}$ as follows:

$$\begin{aligned}
(\boldsymbol{CA})\boldsymbol{E} &= \boldsymbol{E\Gamma} \Rightarrow \boldsymbol{CA} = \boldsymbol{E\Gamma E}^{-1} \\
\Rightarrow [\boldsymbol{I} + 2\Delta\lambda\boldsymbol{CA}]^{-1} &= \boldsymbol{E}[\boldsymbol{I} + 2\Delta\lambda\boldsymbol{\Gamma}]^{-1}\boldsymbol{E}^{-1} = \boldsymbol{EQ}_d^{-1}\boldsymbol{E}^T \\
\boldsymbol{\Gamma} &= diag[\Gamma_1.......\Gamma_6] \qquad \boldsymbol{\Gamma} = \boldsymbol{\Gamma}_A\boldsymbol{\Gamma}_C \\
f = \boldsymbol{\sigma}_{trial}^T\boldsymbol{A}^*\boldsymbol{\sigma}_{trial} &- (1 + \frac{H}{\sigma_y}(\epsilon_{p,n} + 2\Delta\lambda\sqrt{\boldsymbol{\sigma}_{trial}^T\boldsymbol{A}^*\boldsymbol{\sigma}_{trial}}))^2 = 0 \qquad (69) \\
\boldsymbol{A}^* &= \boldsymbol{EQ}_d^{-1}\boldsymbol{\Gamma}_A\boldsymbol{Q}_d^{-1}\boldsymbol{E}^T
\end{aligned}$$

The following matrices are applied in the present implementation:

$$\begin{aligned}
\boldsymbol{\Gamma}_A &= diag[\frac{1}{2n_0^2}, \frac{3}{2n_0^2}, \frac{3}{n_0^2}, \frac{1}{2m_0^2}, \frac{3}{2m_0^2}, \frac{3}{m_0^2}] \\
\boldsymbol{\Gamma}_C &= t * diag[\frac{1}{1-\nu}, \frac{1}{1+\nu}, \frac{1}{2(1+\nu)}, \frac{t^2}{12(1-\nu)}, \frac{t^2}{12(1+\nu)}, \frac{t^2}{24(1+\nu)}] \\
\boldsymbol{E} &= \begin{bmatrix} \boldsymbol{I} & 0 \\ 0 & \boldsymbol{I} \end{bmatrix}\bar{\boldsymbol{E}} \qquad \bar{\boldsymbol{E}} = \frac{\sqrt{2}}{2}\begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix}
\end{aligned}$$

The consistent material tangent for an integration point in the plane is obtained as follows:

$$\begin{aligned}
d\boldsymbol{\epsilon}_{n+1} &= \boldsymbol{C}d\boldsymbol{\sigma}_{n+1} + d\boldsymbol{\epsilon}_{p,n+1} = \boldsymbol{H}^{-1}d\boldsymbol{\sigma}_{n+1} + 2\boldsymbol{A\sigma}_{n+1}d\lambda \\
\Delta\epsilon_p &= 2\Delta\lambda\bar{\sigma} \qquad \bar{\sigma} = \sqrt{\boldsymbol{\sigma}_{n+1}^T\boldsymbol{A\sigma}_{n+1}} \\
df_{n+1} &= 0 \Rightarrow d\lambda = \frac{1}{\beta}2\boldsymbol{\sigma}_{n+1}^T\boldsymbol{A}d\boldsymbol{\sigma}_{n+1} \\
\beta &= \frac{2\alpha\bar{\sigma}}{1 - \alpha\frac{\Delta\lambda}{\bar{\sigma}}} \qquad \alpha = \frac{2H}{\sigma_y^2}(\sigma_y + H\epsilon_{p,n+1}) \\
\Rightarrow [\boldsymbol{H}^{-1} + \frac{1}{\beta}\boldsymbol{gg}^T]d\boldsymbol{\sigma}_{n+1} &= d\boldsymbol{\epsilon}_{n+1} \qquad \boldsymbol{H}^{-1} = \boldsymbol{C}^{-1} + 2\Delta\lambda\boldsymbol{A} \qquad \boldsymbol{g} = 2\boldsymbol{A\sigma} \\
d\boldsymbol{\sigma} &= [\boldsymbol{H} - \frac{\boldsymbol{Hgg}^T\boldsymbol{H}}{\boldsymbol{g}^T\boldsymbol{Hg} + \beta}]d\boldsymbol{\epsilon} = \boldsymbol{C}_t d\boldsymbol{\epsilon} \qquad (70)
\end{aligned}$$

$\boldsymbol{H}$ is expressed in closed form as $\boldsymbol{EQ}_d^{-1}\boldsymbol{E}^T\boldsymbol{C}$.

6. DISCRETISATION AND NUMERICAL SOLUTION

The finite element employed herein is triangular with six dof at each node, i.e. a higher order membrane displacement interpolation (drilling degree of freedom). The procedure for construction of the stiffness is presented by Militello and Felippa[15], but some expressions are listed here for completeness. The element stiffness is split into a basic and higher order contribution. The basic stiffness is derived from a constant stress in the element doing virtual work on element boundary displacements described in terms of the visible degrees of freedom:

$$\mathbf{Kv} = (\mathbf{K_b} + \mathbf{K_h})\mathbf{v} = \mathbf{f} \tag{71}$$

$$\mathbf{K_b} = \frac{1}{V}\mathbf{L}\mathbf{C_e}\mathbf{L^T} \qquad \int_{\mathbf{S}} \delta\mathbf{d^T}\bar{\boldsymbol{\sigma}}_\mathbf{n}\mathbf{dS} = \delta\mathbf{v^T}\int_{\mathbf{S}}\mathbf{N_d^T}\mathbf{T_n}\mathbf{dS}\bar{\boldsymbol{\sigma}} = \delta\mathbf{v^T}\mathbf{L}\bar{\boldsymbol{\sigma}} \tag{72}$$

Here $\bar{\boldsymbol{\sigma}}, \mathbf{d}, \mathbf{L}$ are the constant stress, element boundary displacements, and nodal lumping matrix, respectively. With this approach, the individual element test is satisfied. The higher order stiffness may be derived in many ways, e.g. via the free formulation, assumed natural strain, or assumed natural deviatoric strain (ANDES). The latter is chosen herein, with local (invisible) dof for bending from curvature gages at element midsides and from the drilling dof for the membrane part. The higher order stiffness may be expressed by

$$\mathbf{K}_h = \beta\mathbf{Q}^T\mathbf{K}_d\mathbf{Q} \qquad \mathbf{K_d} = \int_{\mathbf{V}}\mathbf{A_d^T}\mathbf{C_e}\mathbf{A_d}\mathbf{dV} \tag{73}$$

$$\epsilon = \mathbf{Ag} = \mathbf{AQv} = (\bar{\mathbf{A}} + \mathbf{A_d})\mathbf{Qv} \tag{74}$$

$\bar{\mathbf{A}} = (1/V)\int_V \mathbf{AdV}$ is the average assumed strain, hence $\mathbf{A}_d$ is the deviatoric part applied in the higher order terms. The strain dof $\mathbf{g}$ is linked to the visible dof by $\mathbf{Q}$. β is a scaling factor. Four integration points in the plane is used in establishing the element stiffness, located at center and midside points.

The balance equation for internal and external forces for the assembled element model reads

$$\mathbf{r}(\mathbf{v}, \lambda) = \mathbf{f}(\mathbf{v}) - \mathbf{p}(\lambda) = \mathbf{0} \tag{75}$$

The external loads are hence written as a function of a time like parameter λ. Advancing from state n to $n+1$ is carried out by means of a corresponding load increment followed by Newton-Rapshon iterations on the residual. The Riks-Wempner arc length method (normal plane) is employed in order to traverse limit points[24,25]. The iterative corrections to the dof and load increment scaling are given by

$$\delta\mathbf{v}^{k+1} = (\mathbf{K}_{n+1}^k)^{-1}(\frac{\partial\mathbf{p}}{\partial\lambda}\delta\lambda^{k+1} - \mathbf{r}_{n+1}^k) \tag{76}$$

$$\delta\lambda^{k+1} = ((\mathbf{K}_{n+1}^0)^{-1}\frac{\partial\mathbf{p}}{\partial\lambda})^T(\mathbf{K}_{n+1}^k)^{-1}\mathbf{r}_{n+1}^k)/(1 + ((\mathbf{K}_{n+1}^0)^{-1}\frac{\partial\mathbf{p}}{\partial\lambda})^T(\mathbf{K}_{n+1}^k)^{-1}\frac{\partial\mathbf{p}}{\partial\lambda}) \tag{77}$$

The update of the global displacement state is obtained as follows

$$\textit{displacements}: \qquad \mathbf{v} := \mathbf{v} + \Delta\mathbf{v} \tag{78}$$

$$\textit{rotations}: \qquad \mathbf{R} := \mathbf{R}(\Delta\omega)\mathbf{R} \tag{79}$$

7. NUMERICAL SIMULATIONS

In the following, some cases are analysed in order to check the performance of the simplified plasticity model. Cases with loading dominated by either membrane or bending conditions should be accurate (cfr Fig.2), whereas cases with combined load carrying may be nonconservative. Other numerical studies with the present formulation may be found in Ref. 26.

7.1 Plate with uniformly distributed load

The plate is rectangular with length to width ratio 3 and length to thickness 20. All of the plate is modelled, with different mesh refinement. The plate is simply supported on all egdes. The material is modelled as elastic perfectly plastic with yield stress 400 MPa. The characteristics of the response is that five yield lines develop in the plate. A good check of the bending performance of the plasticity model is to compute limit loads for plates. Both upper and lower bound analytic solutions exist for this case. They are very close, indicating that the analytic solution is accurate. In Fig.3 the applied pressure is normalised by the lower bound limit pressure. For the displacement levels plotted, the effects of nonlinear geometry are negligible. Considering the three different levels of mesh refinement, a clear convergence to the limit load is observed. The accuracy of the simulations is considered good for this bending dominated case. One should note that even the finest mesh is rather coarse in order to capture the yield lines in the plate. Simo and Kennedy[6] analysed a quadratic plate with a concentrated load at center with a mesh of much higher density than in the present study.

7.2 Pinched cylinder

In this cases a short cylinder (R=300, L=300, thickness=3) bounded by a rigid diaphragm at each end is loaded by two concentrated forces at midsection. Symmetry allows for modelling of only one octant. A yield stress of 24.3, Young's modulus 3000, and hardening modulus 50 (cfr Brank et al[5]) is employed to model the material (isotropic hardening). Note that this hardening modulus is the slope of a uniaxial stress-strain curve, and is not equal to H used in the stress resultant formulation. Due to the equivalent strain definition for this formulation, H is determined by simulation the tensile test with the stress resultant model. Hence, the uniaxial hardening modulus is applied indirectly. This case represents a complex shell stress distribution, with nonproportional membrane and bending moment histories.

Fig.4 depicts several simulations of different mesh refinements along with two published simulations. Simo and Kennedy[6] employed the complete Ilyushin-Shapiro yield surface (i.e. multisurface model), whereas Brank et al[5] use a Mises material and seven integration points through thickness. All simulations account for large rotations. The simulation by Brank et al should be considered the most accurate (also including out-of-plane shear deformations and change in thickness). For the present coarse mesh simulation, several limit points are observed. This is due to the flat shell modelling leading to several local snap throughs. It is observed that for the finest mesh, the result is located between the two published results. The single limit point in the analysis by Brank et al at a deflection approximately 180 corresponds to a change in global deformation mode. Similar deformations are obtained with the meshes used in the present study. One reason for obtaining the limit load at a vertical displacement of about 160 as opposed to 180 for the accurate analysis, may be due to not accounting for out-of-plane shear deformations. Since the present simulations are somewhat stiffer than the one by Brank et al, it is also possible that the nonconservatism in the yield surface causes this. The simulation with the finest mesh is considered acceptable.

7.3 Collapse of the Scordelis-Lo roof

This case is demanding, showing combined membrane and bending load carrying, and a very nonlinear behaviour. The roof is collapsing under increasing self-weight. It has a geometry as a part of a cylindrical shell (length=2*7.6m, radius=7.6m, part of cylinder angle = 80), and is carried by a rigid diaphragm at each end (the two straight edges are free). One quarter of the roof is modelled. The material has yield stress 4.2MPa, Young's modulus 21000MPa, and is nonhardening. A reference load of $4kN/m^2$ is employed for the gravity. This case has been investigated by several authors, with the most complete by Peric and Owen[10], and Brank et al[5]. In all these simulations integration through thickness with a Mises material was employed. Fig.5 illustrates the simulated response and the finest FE mesh. It is noted that the rapid load decrease in the present study at about 1m corresponds mostly to the simulation by Peric and Owen, wheras the analysis by Brank et al shows a somewhat delayed drop. Comparing the curves denoted 16*16 and 16*16 (12% reduced yield stress), the effect of reducing the yield stress by 12% due to the inaccurate yield surface is observed. It is also noted that the curve corresponding to the modified yield stress has a limit load in good resemblance with the published simulations. Furthermore, the effect of refining the finite element mesh 16*16 is clearly seen.

7.4 Axially loaded plate with snap-through

The plate analysed has the same length to width ratio and boundary conditions as in section 7.1. The load is, however, applied axially. The plate has an initial sinusoidal imperfection of amplitude half of the thickness. The material is a structural steel modelled as isotropic hardening, with yield stress 320MPa and uniaxial hardening modulus 3500MPa. The elastic buckling mode with the lowest eigenvalue has three sinusoidal half-waves in the longitudinal direction and one in the transversal direction. Hence, the plate exhibits a snap through behaviour from one two three longtitudinal waves. This also leads to very nonproportional stress resultant behaviour. Fig.6a shows the axial load versus axial displacement. The ordinate axis is normalised by the Euler load, and the abscissa is normalised by the displacement corresponding to the Euler load. The case was analysed by Soreide[27] with a 9-noded triangle/LST element with 3 midside integration points in the plane and 6 integration points over the thickness. It is noted that the initial elastic stiffness agrees well in the three simulations plotted. The collapse load is overpredicted somewhat by the present analyses. Two explanations may be given. First the lack of capturing gradual plastification over the thickness with the stress resultant yield surface, secondly the yield surface is nonconservative. In buckling problems it is known that accurate modeling of stress and strain over the thickness may be necessary in order to obtain accurate capacities. However, reducing the yield stress as shown in the previous section takes the collapse load close to the result by Soreide. In Ref.26 collapse simulation of a shell without this snap-through (i.e. buckling in one mode) corresponded well with another layer model simulation. Fig.6b illustrates the evolution of the central vertical displacment versus axial displacement. The snap trough is clearly seen. The two curves correspond well.

8. CONCLUDING REMARKS

The present investigation has addressed the performance of a very simple stress resultant plasticity model for thin shell applications in combination with a high performance triangular finite element derived by Militello and Felippa. The co-rotated approach accounts for large rigid body rotations, assuming small strains in the co-rotated co-ordinate system. A complete consistent tangent stiffness in terms of nonlinear geometry and plasticity is derived. In all analyses strict convergence criteria are used (energy norm 10^{-14}, displacement or internal force norm 10^{-7}), showing good convergence in the Newton-Rapshon iterations. The simple yield surface employed is nonconservative (maximum 12%) compared to the traditional Ilyushin criterion. This inaccuracy occurs in combined membrane and bending, otherwise the error vanish. The advantage of the present approach is, in addition to not having to integrate over shell thickness, having a smooth yield surface. This is numerically very beneficial. Although plasticity theory for yield surfaces with corners exists, more calculations at integration point level are required. Furthermore, the analyses are more prone to numerical problems. As the inaccuracy in the present yield criterion is known, it is also possible to do some averaging or scaling of the size of the surface in order to remedy this. The simplest correction is to reduce the yield stress with 6%. Then pure membrane or bending loading yields a 6% conservatism, whereas for balanced bending and membrane loading a 6% nonconservatism is obtained. Putting this model bias into the perspective of uncertainties in external loads, geometry and material properties, boundary conditions etc, it may be quite acceptable. Then the present approach is feasible. If the level of uncertainites of all input quantities is very low, a more accurate approach may be required. The present study is the first to incorporate plasticity in an advanced non-conforming shell finite element (ANDES). This seems to work well. However, investigations with simpler shell elements is under study. Some of the test cases analysed have been compared to other published simulation results. Some of them are based on more detailled plasticity modelling, but of the same level of stringency in the modelling nonlinear geometry. However, with the co-rotated approach, the separation of rigid body motion and deformational motion leads to a conceptually simpler exposition than in the other nonlinear shell theory approaches. As no direct comparison in term of computer resource usage among the different approaches is carried out, no conclusion regarding quantified efficiency may be given here. But this should be studied in the future.

Appendix 1: Connection between co-rotational dof and global dof

$$\begin{bmatrix} \delta_R \mathbf{u_{d1}} \\ \delta_R \boldsymbol{\theta}_{d1} \\ \delta_R \mathbf{u_{d2}} \\ \delta_R \boldsymbol{\theta}_{d2} \\ \vdots \\ \delta_R \mathbf{u_{dN}} \\ \delta_R \boldsymbol{\theta}_{dN} \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{H_1} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{H_2} & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{H_N} \end{bmatrix} \cdot \left(\begin{bmatrix} \mathbf{P_{11}} & \mathbf{0} & \mathbf{P_{12}} & \mathbf{0} & \dots & \mathbf{P_{1N}} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{P_{21}} & \mathbf{0} & \mathbf{P_{22}} & \mathbf{0} & \dots & \mathbf{P_{2N}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{P_{N1}} & \mathbf{0} & \mathbf{P_{N2}} & \mathbf{0} & \dots & \mathbf{P_{NN}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{I} \end{bmatrix} \right.$$
$$\left. + \begin{bmatrix} \mathbf{Spin}\,(\mathbf{x_1^n})\mathbf{G_1} & \mathbf{Spin}\,(\mathbf{x_1^n})\mathbf{G_2} & \dots & \mathbf{Spin}\,(\mathbf{x_1^n})\mathbf{G_N} \\ -\mathbf{G_1} & -\mathbf{G_2} & \dots & -\mathbf{G_N} \\ \mathbf{Spin}\,(\mathbf{x_2^n})\mathbf{G_1} & \mathbf{Spin}\,(\mathbf{x_2^n})\mathbf{G_2} & \dots & \mathbf{Spin}\,(\mathbf{x_2^n})\mathbf{G_N} \\ -\mathbf{G_1} & -\mathbf{G_2} & \dots & -\mathbf{G_N} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{Spin}\,(\mathbf{x_N^n})\mathbf{G_1} & \mathbf{Spin}\,(\mathbf{x_N^n})\mathbf{G_2} & \dots & \mathbf{Spin}\,(\mathbf{x_N^n})\mathbf{G_N} \\ -\mathbf{G_1} & -\mathbf{G_2} & \dots & -\mathbf{G_N} \end{bmatrix} \right) \delta \begin{bmatrix} \mathbf{u_{d1}} \\ \boldsymbol{\theta}_{d1} \\ \mathbf{u_{d2}} \\ \boldsymbol{\theta}_{d2} \\ \vdots \\ \mathbf{u_{dN}} \\ \boldsymbol{\theta}_{dN} \end{bmatrix} \qquad (80)$$

$$\mathbf{P_{ab}} = \left(\delta_{\mathbf{ab}} - \frac{\mathbf{1}}{\mathbf{N}} \right) \mathbf{I} = \begin{cases} \mathbf{I} - \frac{\mathbf{1}}{\mathbf{N}} \mathbf{I} & \textbf{for} \quad a = b \\ -\frac{1}{N} \mathbf{I} & \textbf{for} \quad a \neq b \end{cases} \qquad (81)$$

Rearranging, $\delta_R \mathbf{v_d}$ may be written as

$$\begin{aligned} \delta_R \mathbf{v_d} &= \begin{bmatrix} \mathbf{H_{11}} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{H_{22}} & \dots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & = \mathbf{H_{NN}} \end{bmatrix} \cdot \left(\begin{bmatrix} \mathbf{I} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \dots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{I} \end{bmatrix} \right. \\ &\left. - \begin{bmatrix} \mathbf{P_{T_{11}}} & \mathbf{P_{T_{12}}} & \dots & \mathbf{P_{T_{1N}}} \\ \mathbf{P_{T_{21}}} & \mathbf{P_{T_{22}}} & \dots & \mathbf{P_{T_{2N}}} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{P_{T_{N1}}} & \mathbf{P_{T_{N2}}} & \dots & \mathbf{P_{T_{NN}}} \end{bmatrix} - \begin{bmatrix} \mathbf{S_1} \\ \mathbf{I} \\ \mathbf{S_2} \\ \mathbf{I} \\ \vdots \\ \mathbf{S_N} \\ \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{G_1} & \mathbf{G_2} & \dots & \mathbf{G_N} \end{bmatrix} \right) \delta \mathbf{v} \\ &= \mathbf{H} \left(\mathbf{I} - \mathbf{P_T} - \mathbf{P_R} \right) \delta \mathbf{v} = \mathbf{HP} \delta \mathbf{v} \end{aligned} \qquad (82)$$

$$\mathbf{P_{T_{ab}}} = \begin{bmatrix} \frac{1}{N}\mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \qquad \textbf{and} \qquad \mathbf{H_{aa}} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{H_a} \end{bmatrix} \qquad (83)$$

$$\mathbf{S_a} = \begin{bmatrix} -\mathbf{Spin}\,(\mathbf{x_a^n}) \\ \mathbf{I} \end{bmatrix} \qquad (84)$$

Appendix 2: Consistent geometrical tangent stiffness contributions

The rotational geometric stiffness reads

$$
\begin{aligned}
\mathbf{E^T}\delta\mathbf{T^T}\tilde{\mathbf{P}}^\mathbf{T}\tilde{\mathbf{H}}^\mathbf{T}\tilde{\mathbf{f}}_\mathbf{e} &= \mathbf{E^T}\delta\mathbf{T^T}\tilde{\mathbf{f}} = \mathbf{E^T}\begin{bmatrix} \delta\mathbf{T_n^T} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \delta\mathbf{T_n^T} & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \delta\mathbf{T_n^T} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \delta\mathbf{T_n^T} \end{bmatrix}\begin{bmatrix} \tilde{\mathbf{n}}_1 \\ \tilde{\mathbf{m}}_1 \\ \vdots \\ \tilde{\mathbf{n}}_N \\ \tilde{\mathbf{m}}_N \end{bmatrix} \\
&= \mathbf{E^T}\begin{bmatrix} \delta\mathbf{T_n^T}\tilde{\mathbf{n}}_\mathbf{1} \\ \delta\mathbf{T_n^T}\tilde{\mathbf{m}}_\mathbf{1} \\ \vdots \\ \delta\mathbf{T_n^T}\tilde{\mathbf{n}}_\mathbf{N} \\ \delta\mathbf{T_n^T}\tilde{\mathbf{m}}_\mathbf{N} \end{bmatrix} = \mathbf{E^T}\begin{bmatrix} \mathbf{T_n^T}\mathbf{Spin}\,(\delta\tilde{\boldsymbol{\omega}}_\mathbf{r})\tilde{\mathbf{n}}_\mathbf{1} \\ \mathbf{T_n^T}\mathbf{Spin}\,(\delta\tilde{\boldsymbol{\omega}}_\mathbf{r})\tilde{\mathbf{m}}_\mathbf{1} \\ \vdots \\ \mathbf{T_n^T}\mathbf{Spin}\,(\delta\tilde{\boldsymbol{\omega}}_\mathbf{r})\tilde{\mathbf{n}}_\mathbf{N} \\ \mathbf{T_n^T}\mathbf{Spin}\,(\delta\tilde{\boldsymbol{\omega}}_\mathbf{r})\tilde{\mathbf{m}}_\mathbf{N} \end{bmatrix} \\
&= \mathbf{E^T}\begin{bmatrix} -\mathbf{T_n^T}\mathbf{Spin}\,(\tilde{\mathbf{n}}_\mathbf{1})\delta\tilde{\boldsymbol{\omega}}_\mathbf{r} \\ -\mathbf{T_n^T}\mathbf{Spin}\,(\tilde{\mathbf{m}}_\mathbf{1})\delta\tilde{\boldsymbol{\omega}}_\mathbf{r} \\ \vdots \\ -\mathbf{T_n^T}\mathbf{Spin}\,(\tilde{\mathbf{n}}_\mathbf{N})\delta\tilde{\boldsymbol{\omega}}_\mathbf{r} \\ -\mathbf{T_n^T}\mathbf{Spin}\,(\tilde{\mathbf{m}}_\mathbf{N})\delta\tilde{\boldsymbol{\omega}}_\mathbf{r} \end{bmatrix} = -\mathbf{E^T}\mathbf{T^T}\begin{bmatrix} \mathbf{Spin}\,(\tilde{\mathbf{n}}_\mathbf{1}) \\ \mathbf{Spin}\,(\tilde{\mathbf{m}}_\mathbf{1}) \\ \vdots \\ \mathbf{Spin}\,(\tilde{\mathbf{n}}_\mathbf{N}) \\ \mathbf{Spin}\,(\tilde{\mathbf{m}}_\mathbf{N}) \end{bmatrix}\delta\tilde{\boldsymbol{\omega}}_\mathbf{r} \\
&= -\mathbf{E^T}\mathbf{T^T}\tilde{\mathbf{F}}_\mathbf{nm}\delta\tilde{\boldsymbol{\omega}}_\mathbf{r} = -\mathbf{E^T}\mathbf{T^T}\tilde{\mathbf{F}}_\mathbf{nm}\tilde{\mathbf{G}}\mathbf{T}\mathbf{E}\delta\acute{\mathbf{v}} = \mathbf{K_{GR}}\delta\acute{\mathbf{v}}
\end{aligned} \tag{85}
$$

The following relationships were used in this derivation:

$$
\delta\mathbf{T_n^T} = \mathbf{T_n^T}\mathbf{Spin}\,(\delta\tilde{\boldsymbol{\omega}}_\mathbf{r}) \qquad \mathbf{Spin}\,(\delta\tilde{\boldsymbol{\omega}}_\mathbf{r})\tilde{\mathbf{n}}_\mathbf{a} = -\mathbf{Spin}\,(\tilde{\mathbf{n}}_\mathbf{a})\delta\tilde{\boldsymbol{\omega}}_\mathbf{r} \qquad \delta\tilde{\boldsymbol{\omega}}_\mathbf{r} = \tilde{\mathbf{G}}\delta\tilde{\mathbf{v}} \tag{86}
$$

$$
\tilde{\mathbf{F}}_{nm} = \begin{bmatrix} \mathbf{Spin}\,(\tilde{\mathbf{n}}_\mathbf{1}) \\ \mathbf{Spin}\,(\tilde{\mathbf{m}}_\mathbf{1}) \\ \vdots \\ \mathbf{Spin}\,(\tilde{\mathbf{n}}_\mathbf{N}) \\ \mathbf{Spin}\,(\tilde{\mathbf{m}}_\mathbf{N}) \end{bmatrix} \quad \text{and} \quad \tilde{\mathbf{f}} = \begin{bmatrix} \tilde{\mathbf{n}}_1 \\ \tilde{\mathbf{m}}_1 \\ \vdots \\ \tilde{\mathbf{n}}_N \\ \tilde{\mathbf{m}}_N \end{bmatrix} = \tilde{\mathbf{P}}^T\tilde{\mathbf{H}}^T\tilde{\mathbf{f}}_e \tag{87}
$$

The equilibrium projection geometric term is written

$$
\begin{aligned}
\mathbf{E^T}\mathbf{T^T}\delta_\mathbf{R}\tilde{\mathbf{P}}^\mathbf{T}\tilde{\mathbf{H}}^\mathbf{T}\tilde{\mathbf{f}}_\mathbf{e} &= \mathbf{E^T}\mathbf{T^T}\left(\delta_\mathbf{R}\mathbf{I} - \delta_\mathbf{R}\tilde{\mathbf{P}}_\mathbf{T}^\mathbf{T} - \delta_\mathbf{R}\tilde{\mathbf{P}}_\mathbf{R}^\mathbf{T}\right)\tilde{\mathbf{H}}^\mathbf{T}\tilde{\mathbf{f}}_\mathbf{e} = \mathbf{E^T}\mathbf{T^T}\left(-\delta_\mathbf{R}\tilde{\mathbf{P}}_\mathbf{R}^\mathbf{T}\right)\tilde{\mathbf{H}}^\mathbf{T}\tilde{\mathbf{f}}_\mathbf{e} \\
&= -\mathbf{E^T}\mathbf{T^T}\left(\delta_\mathbf{R}\tilde{\mathbf{G}}^\mathbf{T}\tilde{\mathbf{S}}^\mathbf{T} + \tilde{\mathbf{G}}^\mathbf{T}\delta_\mathbf{R}\tilde{\mathbf{S}}^\mathbf{T}\right)\tilde{\mathbf{H}}^\mathbf{T}\tilde{\mathbf{f}}_\mathbf{e} \\
&= -\mathbf{E^T}\mathbf{T^T}\left(\delta_\mathbf{R}\tilde{\mathbf{G}}^\mathbf{T}\tilde{\mathbf{S}}^\mathbf{T} + \tilde{\mathbf{G}}^\mathbf{T}\delta_\mathbf{R}\tilde{\mathbf{S}}^\mathbf{T}\right)\left(\tilde{\mathbf{f}}_\mathbf{b} + \tilde{\mathbf{f}}_\mathbf{u}\right) \\
&= -\mathbf{E^T}\mathbf{T^T}\left(\left(\delta_\mathbf{R}\tilde{\mathbf{G}}^\mathbf{T}\tilde{\mathbf{S}}^\mathbf{T} + \tilde{\mathbf{G}}^\mathbf{T}\delta_\mathbf{R}\tilde{\mathbf{S}}^\mathbf{T}\right)\tilde{\mathbf{f}}_\mathbf{b} + \left(\delta_\mathbf{R}\tilde{\mathbf{G}}^\mathbf{T}\tilde{\mathbf{S}}^\mathbf{T} + \tilde{\mathbf{G}}^\mathbf{T}\delta_\mathbf{R}\tilde{\mathbf{S}}^\mathbf{T}\right)\tilde{\mathbf{f}}_\mathbf{u}\right) \\
&= -\mathbf{E^T}\mathbf{T^T}\left(\tilde{\mathbf{G}}^\mathbf{T}\delta_\mathbf{R}\tilde{\mathbf{S}}^\mathbf{T}\tilde{\mathbf{f}}_\mathbf{b} + \delta_\mathbf{R}\tilde{\mathbf{P}}^\mathbf{T}\tilde{\mathbf{f}}_\mathbf{u}\right)
\end{aligned} \tag{88}
$$

$\tilde{\mathbf{S}}^T$ represents the rigid body rotation vectors, causing $\tilde{\mathbf{S}}^T\tilde{\mathbf{f}}_b = \mathbf{0}$, because balanced forces do not produce any work on a structure during rigid body displacment or rotation. Furthermore, $\delta\tilde{\mathbf{P}}^T\tilde{\mathbf{f}}_u$ can be neglected because it will be very small when C_{0n} and C_n are close. Eqn. 88 reduces to

$$\begin{aligned}
\mathbf{E^T T^T}\delta\mathbf{\tilde{P}^T\tilde{H}^T\tilde{f}_e} &= -\mathbf{E^T T^T\tilde{G}^T}\delta\mathbf{\tilde{S}^T\tilde{f}_b} = -\mathbf{E^T T^T\tilde{G}^T}\sum_{\mathbf{a=1}}^{\mathbf{N}}\begin{bmatrix}\mathbf{Spin}\left(\delta_{\mathbf{R}}\mathbf{\tilde{x}_a^n}\right) & \mathbf{0}\end{bmatrix}\begin{bmatrix}\tilde{\mathbf{n}}_a \\ \tilde{\mathbf{m}}_a\end{bmatrix} \\
&= -\mathbf{E^T T^T\tilde{G}^T}\sum_{\mathbf{a=1}}^{\mathbf{N}}\mathbf{Spin}\left(\delta_{\mathbf{R}}\mathbf{\tilde{x}_a^n}\right)\mathbf{\tilde{n}_a} = \mathbf{E^T T^T\tilde{G}^T}\sum_{\mathbf{a=1}}^{\mathbf{N}}\mathbf{Spin}\left(\mathbf{\tilde{n}_a}\right)\delta_{\mathbf{R}}\mathbf{\tilde{x}_a^n} \\
&= \mathbf{E^T T^T\tilde{G}^T}\sum_{\mathbf{a=1}}^{\mathbf{N}}\begin{bmatrix}\mathbf{Spin}\left(\mathbf{\tilde{n}_a}\right) & \mathbf{0}\end{bmatrix}\begin{bmatrix}\delta_R\tilde{\mathbf{u}}_{da} \\ \delta_R\tilde{\boldsymbol{\omega}}_{da}\end{bmatrix} \\
&= -\mathbf{E^T T^T\tilde{G}^T\tilde{F}_n^T}\delta_{\mathbf{R}}\mathbf{\tilde{v}_d} = -\mathbf{E^T T^T\tilde{G}^T\tilde{F}_n^T\tilde{P}TE}\delta\mathbf{\acute{v}} = \mathbf{K_{GP}}\delta\mathbf{\acute{v}}
\end{aligned} \tag{89}$$

where

$$\tilde{\mathbf{F}}_n = \begin{bmatrix}\mathbf{Spin}\left(\mathbf{\tilde{n}_1}\right) \\ \mathbf{0} \\ \vdots \\ \mathbf{Spin}\left(\mathbf{\tilde{n}_N}\right) \\ \mathbf{0}\end{bmatrix} \quad \text{and} \quad \tilde{\mathbf{f}}_b = \begin{bmatrix}\tilde{\mathbf{n}}_1 \\ \tilde{\mathbf{m}}_1 \\ \vdots \\ \tilde{\mathbf{n}}_N \\ \tilde{\mathbf{m}}_N\end{bmatrix} = \tilde{\mathbf{P}}^T\tilde{\mathbf{H}}^T\tilde{\mathbf{f}}_e \tag{90}$$

The moment correction term is given by

$$\begin{aligned}
\mathbf{E^T T^T\tilde{P}^T}\delta_{\mathbf{R}}\mathbf{\tilde{H}^T\tilde{f}_e} &= \mathbf{E^T T^T\tilde{P}^T}\begin{bmatrix}\mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \delta_R\tilde{\mathbf{H}}_1^T & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \delta_R\tilde{\mathbf{H}}_N^T\end{bmatrix}\begin{bmatrix}\tilde{\mathbf{n}}_1 \\ \tilde{\mathbf{m}}_1 \\ \vdots \\ \tilde{\mathbf{n}}_N \\ \tilde{\mathbf{m}}_N\end{bmatrix} \\
&= \mathbf{E^T T^T\tilde{P}^T}\begin{bmatrix}\mathbf{0} \\ \delta_R\tilde{\mathbf{H}}_1^T\tilde{\mathbf{m}}_1 \\ \vdots \\ \mathbf{0} \\ \delta_R\tilde{\mathbf{H}}_N^T\tilde{\mathbf{m}}_N\end{bmatrix} = \mathbf{E^T T^T\tilde{P}^T}\begin{bmatrix}\mathbf{0} \\ \tilde{\mathrm{M}}_1\delta_R\tilde{\boldsymbol{\omega}}_{d1} \\ \vdots \\ \mathbf{0} \\ \tilde{\mathrm{M}}_N\delta_R\tilde{\boldsymbol{\omega}}_{dN}\end{bmatrix} \\
&= \mathbf{E^T T^T\tilde{P}^T\tilde{M}\tilde{P}TE}\delta\mathbf{\acute{v}} = \mathbf{K_{GM}}\delta\mathbf{\acute{v}}
\end{aligned} \tag{91}$$

where

$$\tilde{\mathbf{M}} = \begin{bmatrix}\mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \tilde{\mathrm{M}}_1 & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \tilde{\mathrm{M}}_N\end{bmatrix} \tag{92}$$

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Figure captions

Figure 1. Co-rotated formulation.

Figure 2. Ilyushin and simplified plastic interaction surface in MN space.

Figure 3. Simply supported plated subjected to uniform lateral pressure.

Figure 4. Pinched cylinder simulations.

Figure 5. Scordelis-Lo roof collapse simulations.

Figure 6. Axially load plate, a) axial load versus axial displacement, b) transversal deflection versus axial displacement.